

# On the supremum of the tails of normalized sums of independent Rademacher random variables

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**Abstract:** A well-known longstanding conjecture on the supremum of the tails of normalized sums of independent Rademacher random variables is disproved. A related conjecture, also recently disproved, is discussed.

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Let  $\varepsilon := (\varepsilon_1, \varepsilon_2, \dots)$  be the sequence of independent Rademacher random variables (r.v.'s), so that  $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$  for all  $i$ . Consider the “maximal” tail function

$$M(x) := \sup \{ P(a \cdot \varepsilon \geq x) : a \in \Sigma \}$$

of the normalized Rademacher sums  $a \cdot \varepsilon := a_1 \varepsilon_1 + a_2 \varepsilon_2 + \dots$ , where  $x \in \mathbb{R}$  and

$$\Sigma := \{ a = (a_1, a_2, \dots) \in \mathbb{R}^{\mathbb{N}} : a_1^2 + a_2^2 + \dots = 1 \}$$

is the unit sphere in  $\ell^2$ . The behavior of  $P(a \cdot \varepsilon \geq x)$  as a function of  $a$  and  $x$  is very complicated; e.g., Fig. 1 in [6] suggests that even for the fixed  $a = (\underbrace{\frac{1}{10}, \dots, \frac{1}{10}}_{100}, 0, 0, \dots)$ , the tail function  $P(a \cdot \varepsilon \geq \cdot)$  behaves rather erratically.

Yet, a number of features of the maximal tail function  $M$  are known. An easy consequence of the central limit theorem is that

$$M(x) \geq \bar{\Phi}(x) \tag{1}$$

for all  $x \in \mathbb{R}$ , where  $\bar{\Phi}$  is the standard normal tail function. Recently an upper bound on  $M(x)$  was obtained, which is asymptotically equivalent for  $x \rightarrow \infty$  to the lower bound  $\bar{\Phi}(x)$  in (1) — see [4], as well as references therein, including ones to applications in probability/statistics and combinatorics/optimization/operations research.

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A well-known conjecture, apparently due to Edelman [2, 7], has been that for all  $x \in \mathbb{R}$

$$M(x) \stackrel{(?)}{=} M^=(x), \quad (2)$$

where

$$M^=(x) := \sup_{n \in \mathbb{N}} \mathbb{P}(e^{(n)} \cdot \varepsilon \geq x) \quad \text{and} \quad e^{(n)} := (\underbrace{\sqrt{\frac{1}{n}}, \dots, \sqrt{\frac{1}{n}}}_n, 0, 0, \dots),$$

so that  $e^{(n)} \in \Sigma$  for all  $n$  and hence  $M(x) \geq M^=(x)$  for all  $x \in \mathbb{R}$ . Thus, the conjecture could be restated as  $M(x) \stackrel{(?)}{\leq} M^=(x)$  for all  $x \in \mathbb{R}$ .

As apparently most other people working in the area, I had believed this conjecture to be true — until hearing recently that it had been disproved by A. V. Zhubr. Then I wrote to him to request further information. While waiting for a response and being aided by the newly acquired belief that conjecture (2) is false, I quickly happened to find a counterexample to the conjecture, using the following heuristics, which may be of interest to readers. Natural and closest competitors of the “equal-weight” sequences  $a = e^{(n)}$  in terms of having a greater value of  $\mathbb{P}(a \cdot \varepsilon \geq x)$  are sequences  $a$  of the form

$$f^{(n,t)} := (\underbrace{\sqrt{\frac{1-t^2}{n}}, \dots, \sqrt{\frac{1-t^2}{n}}}_n, t, 0, 0, \dots) \in \Sigma \quad \text{for } t \in (0, 1) \quad (3)$$

and  $n \in \mathbb{N}$  — cf. e.g. [5, Proof of Proposition 2], where a conjecture presented in [3] was somewhat similarly disproved; [1, Proof of Theorem 4.2]; and [6, Proof of Theorem 2]. Clearly, for all  $t \in (0, 1)$ ,

$$\mathbb{P}(f^{(n,t)} \cdot \varepsilon \geq x) = \frac{1}{2} \mathbb{P}(e^{(n)} \cdot \varepsilon \geq u) + \frac{1}{2} \mathbb{P}(e^{(n)} \cdot \varepsilon \geq v), \quad (4)$$

where

$$u := \frac{x-t}{\sqrt{1-t^2}} \quad \text{and} \quad v := \frac{x+t}{\sqrt{1-t^2}}. \quad (5)$$

Note that for any  $x \in (0, 1]$  one has  $M(x) = M^=(x) = \mathbb{P}(\varepsilon_1 \geq x) = \frac{1}{2}$ , so that (2) holds. So, as least as far as positive values of  $x$  are concerned, one may assume  $x > 1$ . Note next that the tail of the r.v.  $e^{(n)} \cdot \varepsilon$  decreases fast, just as the corresponding normal tail does (and even faster, in a sense, since it simply vanishes in a neighborhood of  $\infty$ ). Therefore, of the two tail values in (4),  $\mathbb{P}(e^{(n)} \cdot \varepsilon \geq u)$  and  $\mathbb{P}(e^{(n)} \cdot \varepsilon \geq v)$ , the first one is greater, and it may be much greater if  $t \in (0, 1)$  is not too close to 0. Thus, for a given value of  $x$ , to make the tail value  $\mathbb{P}(f^{(n,t)} \cdot \varepsilon \geq x)$  compete with  $M^=(x)$ , one may try to focus on making  $\mathbb{P}(e^{(n)} \cdot \varepsilon \geq u)$  large. Since the inequality  $e^{(n)} \cdot \varepsilon \geq u$  is non-strict, it appears to make sense to choose  $x$  and  $t$  in (5) so that  $u$  be one of the atoms of the distribution of the r.v.  $e^{(n)} \cdot \varepsilon$ . Thus, assume that  $u = k/\sqrt{n}$  for some  $k \in \{0, \dots, n\}$ , whence

$$x = t + \frac{k}{\sqrt{n}} \sqrt{1-t^2}. \quad (6)$$

Similarly, to make  $\sup_{m \in \mathbb{N}} \mathbf{P}(e^{(m)} \cdot \varepsilon \geq x) = M^=(x)$  comparatively small, one may want to choose  $x$  other than the rightmost atom,  $\sqrt{m}$ , of the distribution of the r.v.  $e^{(m)} \cdot \varepsilon$ , for any  $m \in \mathbb{N}$ ; cf. e.g. Lemma 1 in [6] and its (very short) proof therein; moreover then, one may try to choose  $x$  to be about as far away as possible from any of these rightmost atoms,  $\sqrt{m}$ . Thus, it appears to make sense to try the values  $x = \sqrt{\frac{14}{10}}, \sqrt{\frac{24}{10}}, \sqrt{\frac{34}{10}}, \dots$  — each of these values taken together with all small enough natural  $n$ ;  $k \in \{0, \dots, n\}$ ; and the corresponding values of  $t$  found, for each such triple  $(x, n, k)$ , as a root of equation (6).

This approach indeed results in a quadruple  $(x, n, k, t)$  with  $x = \sqrt{\frac{74}{10}} = \sqrt{\frac{37}{5}}$ ,  $n = 10$ ,  $k = 8$ , and  $t = \sqrt{\frac{5}{37}}$ , which disproves conjecture (2):

**Proposition 1.** For  $n = 10$ ,  $x = \sqrt{\frac{37}{5}}$ , and  $t = \sqrt{\frac{5}{37}}$ ,

$$M(x) \geq \mathbf{P}(f^{(n,t)} \cdot \varepsilon \geq x) > M^=(x). \quad (7)$$

In the hindsight, it is hardly a coincidence that for the disproving triple  $(n, x, t) = (10, \sqrt{\frac{37}{5}}, \sqrt{\frac{5}{37}})$  one has  $t = 1/x$ . Indeed, for a given  $u = k/\sqrt{n}$ , it makes sense to choose  $x$  to be as large as possible, in order to make  $M^=(x)$  as small as possible. That is, one can try to take the largest value of  $x$  for which equation (6) has a solution  $t \in (0, 1)$ . Thus, one obtains

$$x = \sqrt{1 + k^2/n} \quad \text{and} \quad t = 1/x.$$

To prove Proposition 1, we shall need the Berry–Esseen inequality

$$\left| \mathbf{P}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \geq x\right) - \bar{\Phi}(x) \right| \leq \frac{C}{\sqrt{n}} \mathbf{E}|X_1|^3 \quad (8)$$

for all  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ , where  $X_1, X_2, \dots$  are independent identically distributed zero-mean unit-variance r.v.'s and  $C$  is an absolute constant. The apparently smallest currently known value of  $C$ , equal to 0.4748, is due to Shevtsova [8]; a slightly larger value, 0.4785, was established earlier by Tyurin [9].

*Proof of Proposition 1.* The first inequality in (7) follows immediately from the definitions of  $M(x)$  and  $f^{(n,t)}$ , since  $f^{(n,t)} \in \Sigma$ . Next, take indeed  $n = 10$ ,  $x = \sqrt{\frac{37}{5}}$ , and  $t = \sqrt{\frac{5}{37}}$ . Then, by (8) with the mentioned constant factor  $C = 0.4748$ , for all natural  $j > 50640$

$$\mathbf{P}(e^{(j)} \cdot \varepsilon \geq x) \leq \bar{\Phi}(x) + \frac{0.4748}{\sqrt{50641}} < \mathbf{P}(f^{(n,t)} \cdot \varepsilon \geq x).$$

On the other hand, it is straightforward to check that

$$\mathbf{P}(f^{(n,t)} \cdot \varepsilon \geq x) > \max_{1 \leq j \leq 50640} \mathbf{P}(e^{(j)} \cdot \varepsilon \geq x)$$

(which takes about 20 minutes to do using Mathematica). Recalling now the definition of  $M^=(x)$ , one obtains the second inequality in (7) as well.  $\square$

Soon after finding the counterexample described above and informing A. V. Zhubr about it, I received a response from him with a copy of his paper [10]. It turns out that in it, not conjecture (2), but the following related “finite-dimensional” conjecture was disproved:

$$M_n(x) \stackrel{(?)}{=} M_n^-(x) \quad (9)$$

for all  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$ , where

$$M_n(x) := \max \{ P(a \cdot \varepsilon \geq x) : a = (a_1, \dots, a_n, 0, 0, \dots) \in \Sigma \} \quad \text{and} \\ M_n^-(x) := \max_{j \in \mathbb{N}, j \leq n} P(e^{(j)} \cdot \varepsilon \geq x).$$

How do conjectures (2) and (9) relate to each other?

To approach an answer to this question, let us start by taking any  $x \in \mathbb{R}$ . Then take any sequence  $a = (a_1, a_2, \dots) \in \Sigma$ . For each large enough  $n \in \mathbb{N}$ , consider the “truncated” version  $a^{[n]} := (a_1, \dots, a_n, 0, 0, \dots) / \sqrt{a_1^2 + \dots + a_n^2}$  of the sequence  $a$ , so that  $a^{[n]} \in \Sigma$  and  $a^{[n]} \cdot \varepsilon \rightarrow a \cdot \varepsilon$  almost surely as  $n \rightarrow \infty$ . Hence, by the Fatou lemma,

$$\liminf_{n \rightarrow \infty} P(a^{[n]} \cdot \varepsilon \geq x) \geq P(a \cdot \varepsilon \geq x). \quad (10)$$

In particular, it follows that

$$M_n(x) \uparrow M(x) \quad (11)$$

for each  $x \in \mathbb{R}$  as  $n \rightarrow \infty$ .

More specifically, suppose now that the real number  $x$  and the sequence  $a = (a_1, a_2, \dots) \in \Sigma$  disprove conjecture (2) in the sense that  $P(a \cdot \varepsilon \geq x) > M^-(x)$  – cf. (7). Then, by (10),  $P(a^{[n]} \cdot \varepsilon \geq x) > M^-(x)$  for all large enough  $n \in \mathbb{N}$ . Therefore and because  $M^-(x) \geq M_n^-(x)$ , it follows that, if conjecture (2) is disproved, then conjecture (9) is disproved as well.

However, we shall now see that the vice versa implication is not true, in the sense that one can find some  $n \in \mathbb{N}$ ,  $a = (a_1, \dots, a_n, 0, 0, \dots) \in \Sigma$ , and  $x \in (0, \infty)$  such that  $P(a \cdot \varepsilon \geq x) > M_n^-(x)$  and yet  $P(a \cdot \varepsilon \geq x) < M^-(x)$ ; in fact, one can find such  $a$  and  $x$  for every large enough  $n \in \mathbb{N}$ .

Indeed, for each natural  $n \geq 8$  let

$$a^{(n)} := f^{(n, t_n)} \quad \text{and} \quad t_n := 1/x_n, \quad \text{where} \quad x_n := \sqrt{n - 3 + 4/n} \quad (12)$$

and  $f^{(n, t)}$  is as in (3); then, as shown in [10, Theorem 3 and Corollary 4],

$$M_n(x_n) \geq P(a^{(n)} \cdot \varepsilon \geq x_n) = (n + 1)/2^{n+1} > M_n^-(x_n),$$

which disproves conjecture (9). This series of counterexamples was obtained in [10] based on certain geometric, rather than probabilistic, considerations (and using rather different notations).

However, for all large enough  $n$  and  $a^{(n)}$  as in (12),

$$\mathbf{P}(a^{(n)} \cdot \varepsilon \geq x_n) < M^-(x_n). \quad (13)$$

Indeed, for  $n \rightarrow \infty$  one has  $\mathbf{P}(a^{(n)} \cdot \varepsilon \geq x_n) = (n+1)/2^{n+1} = 2^{-n(1+o(1))}$ , which is asymptotically much less than  $\overline{\Phi}(x_n) = \exp\{-x_n^2/(2+o(1))\} = (\sqrt{e})^{-n(1+o(1))}$ ; in fact, it is rather easy (even if a bit tedious) to see that  $\mathbf{P}(a^{(n)} \cdot \varepsilon \geq x_n) < \overline{\Phi}(x_n)$  for all natural  $n \geq 16$  and hence (13) holds for such  $n$  — because, again by the central limit theorem (cf. (1)),  $M^-(x) \geq \overline{\Phi}(x)$  for all  $x \in \mathbb{R}$ .

On the other hand — for  $n = 10$ , and  $x_n$  and  $t_n$  as in (12) — one has  $x_n = \sqrt{37/5}$  and hence  $t_n = \sqrt{5/37}$ , so that (as was pointed out to me by A. V. Zhubr) the example given in Proposition 1 turns out to belong to the series of examples defined in (12). Moreover, one can check, just as easily as in the proof of Proposition 1, that (7) will hold with the triple  $(n, x_n, t_n) = (8, \sqrt{11/2}, \sqrt{2/11})$  or  $(n, x_n, t_n) = (9, \sqrt{58/9}, \sqrt{9/58})$  in place of the triple  $(n, x, t) = (10, \sqrt{37/5}, \sqrt{5/37})$  used in Proposition 1. In fact, each of these two checks, for  $n = 8$  and  $n = 9$ , requires much less computer time, respectively about 4 seconds and one minute instead of 20 minutes, since one can then use much smaller threshold values: 3461 for  $n = 8$  and 12775 for  $n = 9$ , in place of the threshold value 50640 for  $n = 10$  used in the proof of Proposition 1. Quite possibly, (7) may similarly hold for some other triples of the form  $(n, x_n, t_n)$  with  $x_n$  and  $t_n$  as in (12) and  $n \in \{11, \dots, 15\}$ ; however, checking that for any one of these 5 values of  $n$  seems to require too much computer time. On the other hand, as was noted, (7) will not hold for any such triples  $(n, x, t) = (n, x_n, t_n)$  with  $n \in \{16, 17, \dots\}$ .

One may also note that, in view of (11), conjecture (2) can be restated in a quasi-finite-dimensional form, as  $\sup_{n \in \mathbb{N}} M_n(x) \stackrel{(?)}{=} M^-(x)$  for all  $x \in \mathbb{R}$ .

Observe that  $M^-(x)$  is the supremum of  $\mathbf{P}(a \cdot \varepsilon \geq x)$  over all sequences  $a \in \Sigma$  taking only one nonzero value. A question that remains open here is whether, for all  $x \in \mathbb{R}$ , the supremum (equal by definition to  $M(x)$ ) of  $\mathbf{P}(a \cdot \varepsilon \geq x)$  over all sequences  $a \in \Sigma$  is the same as that over all sequences  $a \in \Sigma$  taking exactly (or, equivalently, at most) two nonzero values (cf. [10, Theorem 6]). More generally, one may ask whether, for some  $q \in \{2, 3, \dots\}$  and all  $x \in \mathbb{R}$ , the value of  $M(x)$  is equal to the supremum of  $\mathbf{P}(a \cdot \varepsilon \geq x)$  over all sequences  $a \in \Sigma$  taking exactly (or, equivalently, at most)  $q$  nonzero values.

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